Distribution of bipartite entanglement for random pure states

Olivier Giraud

Laboratoire de Physique Théorique, UMR 5152 du CNRS, Université Paul Sabatier, 31062 Toulouse Cedex 4, France

E-mail: giraud@irsamc.ups-tlse.fr

Abstract. We calculate analytic expressions for the distribution of bipartite entanglement for pure random quantum states. All moments of the purity distribution are derived and an asymptotic expansion for the distribution itself is deduced. An approximate expression for moments and distribution of Meyer-Wallach entanglement for random pure states is then obtained.

PACS numbers: 03.67.Mn, 03.67.-a

Introduction

The question of generating and measuring entanglement in multipartite quantum systems has become of greater interest with the development of the field of quantum information. Entanglement generation is an important aspect of several quantum information processes, such as superdense coding [1], quantum communication [2], or quantum data hiding [3]. Various methods have been proposed in order to generate highly entangled quantum states, based on pseudo-random unitary operators [4] or on the entangling power of chaotic quantum maps [5, 6] or intermediate quantum maps [7]. Entanglement generation by means of pseudo-random unitary operators or chaotic quantum maps relies upon the fact that unitary evolution of any initial state leads to states whose entanglement properties are close to those of random states, in particular to highly entangled states.

In order to quantify the entanglement of a state, or the entangling power of an operator, a number of entanglement measures have been proposed, based either on quantum information theory or on thermodynamical considerations: entanglement of formation and distillable entanglement [8], relative entropy [9, 10, 11], n-tangle [12], concurrence [13]. For bipartite entanglement of pure states, these measures all reduce to the entropy of entanglement [14], which can be proved to be a unique entanglement measure [15, 16]. The entropy of entanglement corresponds to the von Neumann entropy of the partial density matrix obtained by tracing over one subsystem. Rather than the von Neumann entropy itself, one often prefers to consider the purity R, which corresponds (up to constants) to the so-called linear entropy, that is the first-order term in the expansion of the von Neumann entropy around its maximum. To quantify the degree of entanglement of multipartite pure states, one measure commonly used, based on purity, is the measure proposed by Meyer and Wallach in [17]. It consists in taking the average of the bipartite entanglement of one qubit with all others, measured by the purity (see Equation (18)) [18]. Meyer-Wallach entanglement was used e.g. to quantify entanglement generation for pseudo-random operators [4] or intermediate or chaotic quantum maps [7, 19, 20].

The study of purity or Meyer-Wallach entanglement is of particular interest for random quantum states. Random pure states as column vectors of random unitary matrices distributed according to the invariant Haar measure can be shown to be entangled with high probability. Various analytical calculations have been carried out to characterize entanglement properties of random states. Expressions for the first moment of the purity have been obtained by Lubkin [21]; the second and third moments have been derived in [6], following earlier work [23]. The average entropy has been obtained in [22]. Statistical properties of entanglement measures for random density matrices were obtained in [25]-[27]. The average value for each Schmidt coefficient of a random pure state has been calculated in [28]. In [29], the average entropy of a subsystem was obtained from the average Tsallis entropy [30].

To further characterize entanglement of random pure states, our aim here is to give an exact expression for all moments of the probability density distribution $P(R)$ of the purity for a bipartite random pure state. Since the probability distribution $P(R)$ is defined over a bounded interval (R is bounded), the knowledge of all moments determines uniquely the probability distribution [31]. There are various techniques to obtain a function approximating the exact probability density distribution in a controlled way (that is, by an expansion where the error can be bounded) from the knowledge of its moments. In [32] an algorithm was given to construct polynomials

converging to the probability distribution. We will rather follow [33], where the asymptotic expansion for nearly gaussian distributions is calculated at all orders. In Section 1 the moments $\langle R^n \rangle$ for the distribution $P(R)$ are calculated, and the construction of the asymptotic expansion of $P(R)$ at all orders from its moments is recalled. In Section 2 the approximate moments $\langle Q^n \rangle$ for the distribution $P(Q)$ are derived. For both distributions, the moments are expressed as sums involving a finite number of combinatorial terms and can be easily calculated effectively. As an illustration, we give the first values of the cumulants and calculate the probability density distribution expansion for Meyer-Wallach entanglement.

1. Bipartite entanglement for random pure states

Let Ψ be a pure state belonging to a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are spanned respectively by $\{|a_i\rangle\}_{1\leq i\leq p}$ and $\{|b_i\rangle\}_{1\leq i\leq q}$. We assume that $p \leq q$. Let x_i be the Schmidt coefficients for Ψ. That is, the state Ψ has a Schmidt decomposition (see e.g. [34])

$$|\Psi\rangle = \sum_{i=1}^{p} \sqrt{x_i}|a_i\rangle \otimes |b_i\rangle. \tag{1}$$

The bipartite entanglement measure for Ψ can be expressed throught Schmidt coefficients x_i. The entropy of entanglement is the Shannon entropy of the x_i's: $S(\Psi) = -\sum_{i=1}^{p} x_i \ln x_i$. The purity $R(\Psi)$ of the state Ψ can be expressed as

$$R(\Psi) = \sum_{i=1}^{p} x_i^2. \tag{2}$$

For random states the Schmidt coefficients are distributed according to the density

$$P(x_1, \ldots, x_p) = \mathcal{N} \prod_{1\leq i<j\leq p} (x_i - x_j)^2 \prod_{1\leq k\leq p} x_k^{q-p}\, \delta\left(1 - \sum_{i=1}^{p} x_i\right) \tag{3}$$

for $x_i \in [0, 1]$, with some normalisation factor $\mathcal{N}$ [37, 6]. The n-th moment of the purity is then given by

$$\begin{aligned} \langle R^n \rangle = \mathcal{N} \int_0^1 dx_1 \ldots dx_p \prod_{1\leq i<j\leq p} (x_i - x_j)^2 \prod_{1\leq k\leq p} x_k^{q-p} \\ \times \left(x_1^2 + x_2^2 + \cdots + x_p^2\right)^n \delta\left(1 - \sum_{i=1}^{p} x_i\right). \end{aligned} \tag{4}$$

The calculation of $\langle R^n \rangle$ requires the evaluation of integrals of the form

$$I(\mathbf{n}) = \int_0^1 dx_1 \ldots dx_p V(\mathbf{x})^2 x_1^r \ldots x_p^r \delta\left(1 - \sum_{i=1}^{p} x_i\right) f_{\mathbf{n}}(\mathbf{x}), \tag{5}$$

where $r = q - p$, $\mathbf{x} = (x_1, \ldots, x_p)$ and $\mathbf{n} = (n_1, \ldots, n_p)$. The function $f_{\mathbf{n}}(\mathbf{x}) = \{x_1^{n_1} x_2^{n_2} \ldots x_p^{n_p}$ + all permutations of the $n_i\}$ is a symmetric function of the x_i, and V is the Vandermonde determinant

$$V(\mathbf{x}) = \prod_{1\leq i<j\leq p} (x_i - x_j). \tag{6}$$

The integral $I(\mathbf{n})$ is evaluated in the appendix and yields

$$I(\mathbf{n}) = \frac{p!\prod_{i=1}^{p}(r+n_i+i-1)!}{(p^2+rp+\sum_i n_i-1)!}\prod_{i<j}(n_j-n_i+j-i) + \text{perm.}, \tag{7}$$

where "+perm" indicates that the expression (7) is a sum over all permutations of the n_i. Now the function f for a given $\langle R^n\rangle$ is obtained by multinomial expansion of the term

$$(x_1^2+\cdots+x_p^2)^n = \sum_{n_1+n_2+\cdots+n_p=n}\frac{n!}{n_1!n_2!\dots n_p!}x_1^{2n_1}x_2^{2n_2}\dots x_p^{2n_p}. \tag{8}$$

The normalization constant $\mathcal{N}$ in (4) is given by the choice $\mathbf{n}=\mathbf{0}$ in (5), i.e. $\mathcal{N}=1/I(\mathbf{0})$. This leads to

$$\langle R^n\rangle = \frac{(p^2+rp-1)!}{(p^2+rp+2n-1)!}\sum_{n_1+n_2+\cdots+n_p=n}\frac{n!}{n_1!n_2!\dots n_p!} \times\frac{\prod_{i=1}^{p}(r+2n_i+i-1)!}{\prod_{i=1}^{p}(r+i-1)!}\prod_{1\le i<j\le p}\frac{2n_j-2n_i+j-i}{j-i}. \tag{9}$$

Replacing r by its value $q-p$ and correspondingly changing all indices i to $p+1-i$ (and n_i to n_{p+1-i}), one finally obtains

$$\langle R^n\rangle = \frac{(pq-1)!}{(pq+2n-1)!}\sum_{n_1+n_2+\cdots+n_p=n}\frac{n!}{n_1!n_2!\dots n_p!} \times\prod_{i=1}^{p}\frac{(q+2n_i-i)!}{(q-i)!i!}\prod_{1\le i<j\le p}(2n_i-i-2n_j+j). \tag{10}$$

Note that one can cast (10) into an expression more symmetric in p and q by noting that

$$\frac{\prod_{i<j}(2n_i-i-2n_j+j)}{\prod_{i=1}^{p}(p+2n_i-i)!} = \prod_{j=1}^{p}\left[\frac{1}{(2n_j)!}\prod_{i=1}^{j-1}\left(1-\frac{2n_j}{2n_i+j-i}\right)\right], \tag{11}$$

yielding

$$\langle R^n\rangle = \frac{(pq-1)!}{(pq+2n-1)!}\sum_{n_1+n_2+\cdots+n_p=n}\frac{n!}{n_1!n_2!\dots n_p!} \tag{12}$$
$$\times\prod_{n_i\neq 0}\left[\frac{(q+2n_i-i)!(p+2n_i-i)!}{(q-i)!(p-i)!(2n_i)!}\prod_{j=1}^{i-1}\left(1-\frac{2n_j}{2n_i+j-i}\right)\right].$$

Equation (10) is a closed expression, involving only a finite sum over partitions of n into numbers greater or equal to 0. Note that the order of the n_i matters: for instance for $p=2$ and $n=2$ the sum will involve three terms $(n_1,n_2)=(2,0)$, $(1,1)$ and $(0,2)$. These partitions can be easily generated for any n by some suitable algorithm (see e.g. [33] for such an algorithm generating the partitions required). From Equation (10) one can get the expressions for the cumulants of the distribution $P(R)$. Indeed, given the moments μ_n of a distribution the n-th cumulant κ_n reads (see e.g. [33])

$$\kappa_n = n!\sum_{\{k_m\}}(-1)^{r-1}(r-1)!\prod_{m=1}^{n}\frac{1}{k_m!}\left(\frac{\mu_m}{m!}\right)^{k_m}, \tag{13}$$

where $r = k_1 + \cdots + k_n$, and the sum runs over all $k_i \geq 0, 1 \leq i \leq n$ such that $k_1 + 2k_2 + ... + nk_n = n$. As an example, the first five cumulants read

$$\kappa_1 = \frac{p+q}{1+pq} \tag{14}$$
$$\kappa_2 = \frac{2(p^2-1)(q^2-1)}{(1+pq)^2(2+pq)(3+pq)}$$
$$\kappa_3 = \frac{8(p^2-1)(q^2-1)(p+q)(-5+pq)}{(1+pq)^3(2+pq)(3+pq)(4+pq)(5+pq)}$$
$$\kappa_4 = \frac{48(p^2-1)(q^2-1)(pq-3)A_{p,q}}{(1+pq)^3(2+pq)(3+pq)\prod_{i=1}^{7}(i+pq)}$$
$$\kappa_5 = \frac{384(p^2-1)(q^2-1)(p+q)B_{p,q}}{(1+pq)^4(2+pq)(3+pq)\prod_{i=1}^{9}(i+pq)}$$

where $A_{p,q}$ and $B_{p,q}$ are polynomials in p and q defined by $A_{p,q} = 28-112p^2-153pq-79p^3q-112q^2-98p^2q^2-11p^4q^2-79pq^3-3p^3q^3+p^5q^3-11p^2q^4+4p^4q^4+p^3q^5$ and $B_{p,q} = 3528-6552p^2-6343pq-449p^3q-6552q^2+1545p^2q^2+1237p^4q^2-449pq^3+1164p^3q^3+132p^5q^3+1237p^2q^4-274p^4q^4-41p^6q^4+132p^3q^5-93p^5q^5+p^7q^5-41p^4q^6+9p^6q^6+p^5q^7$. As expected, κ_1 corresponds to Lubkin's expression [21] for the average purity. For $n = 2, 3$ one recovers the expressions derived in [6]. For larger n it is easy to generate the exact value for each cumulant.

In the case $p = 2$, the analytic expression for the probability distribution $P(R)\, dR$ can easily be obtained analytically directly from (2)-(3). It reads

$$P(R)\, dR = A(1-R)^{q-2}\sqrt{2R-1}\, dR \tag{15}$$

for $1/2 \leq R \leq 1$, 0 otherwise (A is the normalization factor). For $p \geq 3$, the asymptotic expansion of the distribution can be obtained (see [33] and references therein) by Edgeworth expansion as a function of the normal distribution $Z(x) = \exp(-x^2/2)/\sqrt{2\pi}$, the mean $\mu = \kappa_1$, the variance $\sigma^2 = \mu_2 - \mu_1^2 = \kappa_2$, and rescaled cumulants $\gamma_r = \kappa_r/\sigma^{2r-2}$:

$$P(R) = \frac{1}{\sigma} Z\left(\frac{R-\mu}{\sigma}\right)\Bigg[1+ \sum_{s=1}^{\infty}\sigma^s \sum_{\{k_m\}} \mathrm{He}_{s+2t}\left(\frac{R-\mu}{\sigma}\right)\prod_{m=1}^{s}\frac{1}{k_m!}\left(\frac{\gamma_{m+2}}{(m+2)!}\right)^{k_m}\Bigg]. \tag{16}$$

For each s the sum runs over $k_j \geq 0$ such that $\sum_j jk_j = s$, and t is defined by $t = \sum_j k_j$. The $\mathrm{He}_n(x)$ are Chebyshev-Hermite polynomials defined by $\mathrm{He}_n(x) = (-1)^n e^{x^2/2}\partial^n e^{-x^2/2}$ (here ∂ is the differential operator with respect to x) and correspond to rescaled Hermite polynomials:

$$\mathrm{He}_n(x) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k x^{n-2k}}{k!(n-2k)!2^k}. \tag{17}$$

Equations (13)-(17) together with the knowledge of the moments (10) allow to obtain explicitely the asymptotic expansion of the probability density distribution at any order.

2. Multipartite entanglement

The Meyer-Wallach entanglement of a pure M-dimensional state Ψ coded on m qubits (with $M = 2^m$) can be defined by

$$Q(\Psi) = 2\left(1 - \frac{1}{m}\sum_{i=1}^{m} R_k\right), \tag{18}$$

where R_k is the purity (2) of the k-th qubit [18]. In order to calculate $Q(\Psi)$ for bipartite random pure states we need to calculate the average purity of a bipartite system belonging to a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ has dimension $p = 2$ and $\mathcal{H}_B$ has dimension $q = 2^{m-1} = M/2$. The moments $\langle R^n \rangle$ can be obtained in this case either from Equation (10) or directly from the distribution (15). In both cases it leads to

$$\langle R^n \rangle = \frac{\Gamma(q+\frac{1}{2})}{\sqrt{\pi}2^{n-1}} \sum_{k=0}^{n} \binom{n}{k} \frac{\left(k+\frac{1}{2}\right)!}{\left(q+k-\frac{1}{2}\right)!}. \tag{19}$$

The calculation of the moments $\langle Q^n \rangle$ involves terms of the form $\langle (\sum_i R_i)^k \rangle$. These terms depend on correlations between the purities R_i. However if we make the assumption that for two different qubits $i \neq j$ we have $\langle R_i R_j \rangle = \langle R_i \rangle \langle R_j \rangle$, we get a distribution $P(Q)$ which turns out to be very close to the numerical distribution obtained by generating random matrices. Making this assumption we get

$$\langle \left(\sum_{i=1}^{m} R_i\right)^k \rangle = \sum_{k_1+k_2+\dots+k_m=k} \frac{k!}{k_1!k_2!\dots k_m!} \langle R^{k_1} \rangle \langle R^{k_2} \rangle \dots \langle R^{k_m} \rangle, \tag{20}$$

The n-th moment is then

$$\langle Q^n \rangle = 2^n \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k k!}{m^k} \sum_{k_1+k_2+\dots+k_m=k} \frac{\langle R^{k_1} \rangle}{k_1!} \frac{\langle R^{k_2} \rangle}{k_2!} \dots \frac{\langle R^{k_m} \rangle}{k_m!}. \tag{21}$$

Gathering together terms having the same exponents, we finally get

$$\langle Q^n \rangle = 2^n \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k k!}{m^k} \sum_{\{r_k\}} \frac{m!}{r_1!r_2!\dots r_k!(m-r)!} \prod_{i=1}^{k} \left(\frac{\langle R^i \rangle}{i!}\right)^{r_i}, \tag{22}$$

where $r \equiv \sum r_i$ and $\langle R^n \rangle$ is given by (19). The sum runs over all $r_i \geq 0$ such that $\sum_j j r_j = k$. From Equations (19) and (22) one can now obtain the cumulants for the distribution $P(Q)$ for an m-qubit system ($M = 2^m$). The first ones read

$$\begin{aligned}
\kappa_1^Q &= \frac{M-2}{M+1} \\
\kappa_2^Q &= \frac{6(M-2)}{(M+1)^2(M+3)m} \\
\kappa_3^Q &= \frac{24(-M^2+7M-10)}{(M+1)^3(M+3)(M+5)m^2} \\
\kappa_4^Q &= \frac{144(M^4-12M^3+6M^2+133M-210)}{(M+1)^4(M+3)^2(M+5)(M+7)m^3} \\
\kappa_5^Q &= -\frac{1152(1890-1763M+337M^2+78M^3-23M^4+M^5)}{(M+1)^5(M+3)^2(M+5)(M+7)(M+9)m^4}.
\end{aligned} \tag{23}$$

We can use these approximate cumulants to obtain an analytical formula for $P(Q)$. Calculating the first terms in the asymptotic expansion (16) we obtain (the first terms can be found in [38])

$$\begin{aligned} P(Q) \sim & \frac{1}{\sigma} Z\left(\frac{Q-\mu}{\sigma}\right)\left\{1+\frac{\tau_3}{6}\mathrm{He}_3\left(\frac{Q-\mu}{\sigma}\right)\right. \\ + & \left[\frac{\tau_4}{24}\mathrm{He}_4\left(\frac{Q-\mu}{\sigma}\right)+\frac{\tau_3^2}{72}\mathrm{He}_6\left(\frac{Q-\mu}{\sigma}\right)\right] \\ + & \left[\frac{\tau_5}{5!}\mathrm{He}_5\left(\frac{Q-\mu}{\sigma}\right)+\frac{\tau_3\tau_4}{144}\mathrm{He}_7\left(\frac{Q-\mu}{\sigma}\right)\right. \qquad (24) \\ + & \left.\left.\frac{\tau_3^3}{1296}\mathrm{He}_9\left(\frac{Q-\mu}{\sigma}\right)\right]+\ldots\right\} \qquad (25) \end{aligned}$$

with $\mu = \kappa_1^Q$, $\sigma = \sqrt{\kappa_2^Q}$ and $\tau_i = \kappa_i/\sigma^i$, $i \geq 3$. Figure 1 displays the probability density function $P(Q)$ for $m = 10$ qubits as obtained by averaging over numerically generated random matrices, together with the plot of analytical expression (24) truncated at order 0 (gaussian), 1 (first line of (24)), 2 (two first lines of (24)) and 3 (expression (24))), using the cumulants (23). The tails of the distribution are reproduced with increasing accuracy when the number of terms in the analytic expansion is increased. Figure 2 displays the same for $m = 11$.

It is to be noted that techniques similar to those used to derive $\langle R^n \rangle$ and $P(R)$ in section 1 can be applied to derive distributions for random states drawn from orthogonal or symplectic matrix ensembles, since the joint probability distribution for Schmidt coefficients is of the same form as the distribution (3).

The author thanks CalMiP in Toulouse and Idris in Orsay for access to their supercomputers, and Bertrand Georgeot for reading the manuscript. This work was supported by the Agence Nationale de la Recherche (ANR project INFOSYSQQ) and the European program EC IST FP6-015708 EuroSQIP.

Appendix A.

The aim of this appendix is to evaluate integrals of the form $I(\mathbf{n})$ given by Equation (5). The Vandermonde determinant (6) can be written

$$V(\mathbf{x}) = \sum_{\sigma} \epsilon_\sigma x_{\sigma(1)}^0 \ldots x_{\sigma(p)}^{p-1}, \qquad (A.1)$$

where the sum runs over all permutations on p elements and ϵ_σ is the signature of the permutation σ. For any function φ symmetric under permutations of the x_i we have

$$\begin{aligned} & \int_0^1 dx_1 \ldots dx_p V(\mathbf{x})^2 \varphi(\mathbf{x}) = \qquad (A.2) \\ & = \sum_{\sigma,\sigma'} \epsilon_\sigma \epsilon_{\sigma'} \int_0^1 dx_1 \ldots dx_p x_{\sigma(1)}^0 x_{\sigma'(1)}^0 \ldots x_{\sigma(p)}^{p-1} x_{\sigma'(p)}^{p-1} \varphi(\mathbf{x}) \\ & = \sum_{\sigma,\sigma'} \epsilon_{\sigma\circ\sigma'} \int_0^1 dx_1 \ldots dx_p x_1^0 x_{\sigma\circ\sigma'(1)}^0 \ldots x_p^{p-1} x_{\sigma\circ\sigma'(p)}^{p-1} \varphi(\mathbf{x}) \\ & = p! \int_0^1 dx_1 \ldots dx_p x_1^0 x_2^1 \ldots x_p^{p-1} V(\mathbf{x}) \varphi(\mathbf{x}), \end{aligned}$$

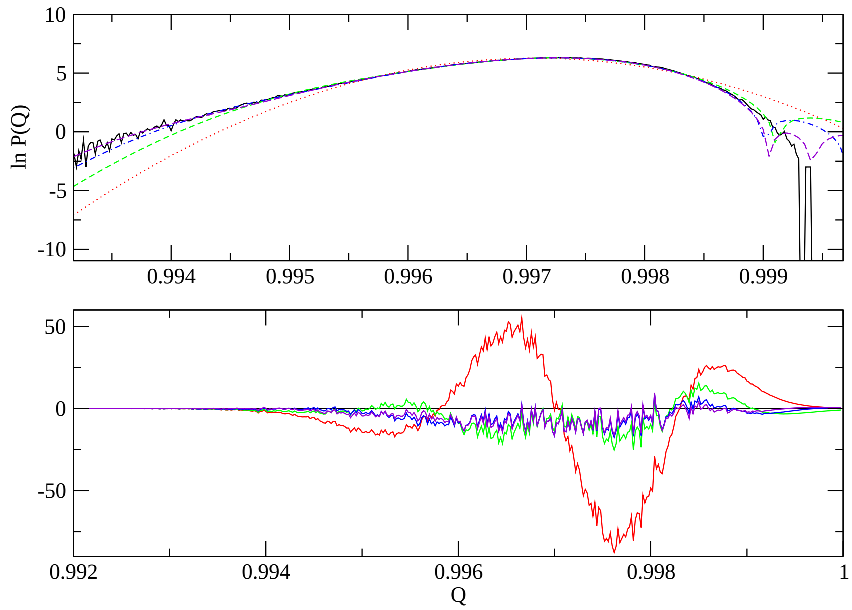

Figure 1. Probability density function $P(Q)$ of Meyer-Wallach entanglement Q for random vectors of size 2^m for $m = 10$. Top: $P(Q)$ in logarithmic scale. Bottom: differences between $P_s(Q)$ (analytical expansion at order s) and the numerical curve. From bottom to top on the left axis of the top figure: truncation of the expansion at order 0 (gaussian, red, dotted); order 1 (green, short dashed); order 2 (blue, dot-dashed); order 3 (purple, long dashed); numerical curve from column vectors of 1000 random unitary matrices obtained by Hurwitz parametrization (black, solid).

with $\mathbf{x} = (x_1, \ldots, x_p)$. The integral (5) becomes $I(\mathbf{n}) = \sum_\tau J(\tau(\mathbf{n}))$ where the sum runs over all permutations of the n_i, with

$$J(\mathbf{n}) = p! \int_0^1 dx_1 \ldots dx_p V(\mathbf{x}) \prod_{i=1}^{p} x_i^{r+n_i+i-1} \delta\left(1 - \sum_{i=1}^{p} x_i\right). \quad \text{(A.3)}$$

Using the fact that

$$\int_0^1 dx x^a (1-x)^b = \frac{a!b!}{(a+b+1)!}, \quad \text{(A.4)}$$

a recurrence on the number of integrals shows that

$$\int_0^1 dx_1 \ldots dx_p x_1^{a_1} \ldots x_p^{a_p} \delta\left(1 - \sum_{i=1}^{p} x_i\right) = \frac{a_1! a_2! \ldots a_p!}{(\sum_{i=1}^{p} a_i + p - 1)!}. \quad \text{(A.5)}$$

The Vandermonde determinant (A.1) can be written as

$$V(\mathbf{x}) = \sum_\sigma \epsilon_\sigma x_1^{\sigma(1)-1} \ldots x_p^{\sigma(p)-1}. \quad \text{(A.6)}$$

Inserting this expression in the integral (A.3) leads to a sum of integrals of the form (A.5), which can be cast under

$$J(\mathbf{n}) = \frac{p! \prod_{i=1}^{p} (r + n_i + i - 1)!}{(p^2 + rp + \sum_i n_i - 1)!} \Delta(\mathbf{n}) \quad \text{(A.7)}$$

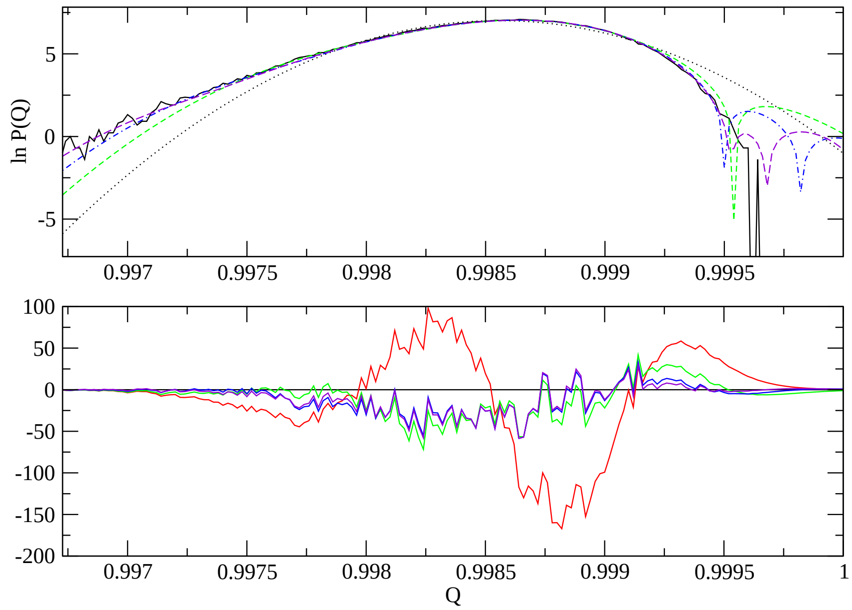

Figure 2. Same as Figure 1 for $m = 11$. Numerical curve averaged over 100 random unitary matrices.

with $\Delta(\mathbf{n})$ a determinant defined by

$$\Delta(\mathbf{n}) = \begin{vmatrix} 1 & r+n_1+1 & (r+n_1+1)(r+n_1+2) & \cdots \\ 1 & r+n_2+2 & (r+n_2+2)(r+n_2+3) & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 1 & r+n_p+p & (r+n_p+p)(r+n_p+p+1) & \cdots \end{vmatrix}. \quad \text{(A.8)}$$

The determinant can be evaluated by recurrence. It finally yields

$$J(\mathbf{n}) = \frac{p! \prod_{i=1}^{p} (r+n_i+i-1)!}{(p^2+rp+\sum_i n_i - 1)!} \prod_{i<j} (n_j - n_i + j - i) \quad \text{(A.9)}$$

which in turn gives $I(\mathbf{n})$ as a sum over permutations of the n_i of $J(\mathbf{n})$.

[1] A. Harrow, P. Hayden, and D. Leung, Phys. Rev. Lett. **92**, 187901 (2004).
[2] P. Hayden, D. Leung, P. Shor, and A. Winter, Commun. Math. Phys. **250**, 371 (2004).
[3] D. P. DiVincenzo, D. W. Leung, and B. M. Terhal, IEEE Trans. Inf. Theory **48** (3), 580-598 (2002).
[4] Y. S. Weinstein and C. S. Hellberg, Phys. Rev. Lett. **95**, 030501 (2005); Y. S. Weinstein and C. S. Hellberg, Phys. Rev. A **71**, 014303 (2005); Y. S. Weinstein and C. S. Hellberg, Phys. Rev. A **69**, 062301 (2004); Y. S. Weinstein and C. S. Hellberg, Phys. Rev. A **72**, 022331 (2005).
[5] J. N. Bandyopadhyay and A. Lakshminarayan, Phys. Rev. Lett. **89**, 060402 (2002).
[6] A.J. Scott and C.M. Caves, J. Phys. A: Math. Gen. **36**, 9553-9576 (2003).
[7] O. Giraud and B. Georgeot, Phys. Rev. A **72**, 042312 (2005).
[8] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).
[9] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. **78**, 2275 (1997).
[10] V. Vedral, M. B. Plenio, K. Jacobs, and P. L. Knight, Phys. Rev. A **56**, 4452 (1997).

[11] V. Vedral and M. B. Plenio, Phys. Rev. A **57** 1619, (1998).
[12] A. Wong and N. Christensen, Phys. Rev. A **63** 044301, (2001).
[13] S. Hill and W. K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997).
[14] C. H. Bennett, H. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).
[15] S. Popescu and D. Rohrlich, Phys. Rev. A **56**, R3319 (1997).
[16] M. J. Donald, M. Horodecki and O. Rudolph , J. Math. Phys. **43**, 4252 (2002).
[17] A. D. Meyer and N. R. Wallach, J. Math. Phys. **43**, 4273 (2002).
[18] G. K. Brennen, Quant. Inf. Comp. **3** 619 (2003).
[19] A.J. Scott, Physical Review A **69**, 052330 (2004).
[20] P. Zanardi, C. Zalka, and L. Faoro, Phys. Rev. A **62**, 030301 (2000).
[21] E. Lubkin, J. Math. Phys. (N.Y.) **19**, 1028 (1978).
[22] D. N. Page, Phys. Rev. Lett. **71**, 1291 (1993).
[23] S. Sen, Phys. Rev. Lett. **77**, 1 (1996).
[24] K. Zyczkowski and H. -J. Sommers , J. Phys. A: Math. Gen. **34**, 7111-7125 (2001).
[25] H. -J. Sommers and K. Zyczkowski, J. Phys. A: Math. Gen. **37**, 8457-8466 (2004).
[26] K. Zyczkowski, 0606228.
[27] V. Cappellini, H. -J. Sommers and K. Zyczkowski, http://www.citebase.org/abstract?id=oai:arXiv.org:quant-ph/0605251 (2006).
[28] M. Znidaric, quant-ph/0611226.
[29] L. C. Malacarne, R. S. Mendes, and E. K. Lenzi, Phys. Rev. E **65**, 046131 (2002).
[30] C. Tsallis, J. Stat. Phys. **52** 479 (1988).
[31] F. Hausdorff, Math. Zeitschr. **9**, 74-109 and 280-299 (1921).
[32] G. Greaves, Numer. Math. **39**, 231-238 (1982).
[33] S. Blinnikov and R. Moessner, Astron. Astrophys. Suppl. Ser. **130** 193-205 (1998).
[34] M. A. Nielsen and I. L. Chuang, Quantum computation and quantum information, Cambridge university press (2000).
[35] G. Vidal, J. Mod. Opt. **47** 355-376 (2000).
[36] M. L. Mehta, Random Matrices (Academic Press, New York, 1991).
[37] S. Lloyd and H. Pagels, Ann. Phys., NY **188**, 186 (1988).
[38] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, Inc, New York (1965).